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QUANTITY AND QUANTITY VALUE

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Abstract – The concept system around 'quantity' and 'quantity value' is fundamental for measurement science, but some very basic issues are still open on such concepts and their relations. This paper proposes a duality between quantities and quantity values, a proposal that simplifies their characterization and makes it consistent.

Keywords: measurement science, quantity, quantity value

1. INTRODUCTION

The concept system around 'quantity' and 'quantity value' is fundamental for measurement science. According to the *International Vocabulary of Metrology – Basic and general concepts and associated terms* (VIM3) [1], indeed, for example measurement units and measurands are quantities, and true values and indications are quantity values. Furthermore, measurement, the "process of experimentally obtaining one or more quantity values that can reasonably be attributed to a quantity" [1, 2.1], is the operational connector of the two concepts.

There are some clues that the relation between 'quantity' and 'quantity value' is still open for discussion and better clarification. Just to remain in the context of JCGM documents, [2] states that "in future editions of [the VIM] it is intended to make a clear distinction between the use of the term error as a quantity and as a quantity value. The same statement applies to the term indication. In the current document such a distinction is made. [On the other hand, the VIM3] does not distinguish explicitly between these uses". The reference is plausibly to some related inconsistencies in the VIM3, as when it is stated that "indications, corrections and influence quantities can be input quantities in a measurement model" [VIM3, 2.50 Note 2], thus assuming indications as quantities, against the definition of 'indication' as "quantity value provided by a measuring instrument or a measuring system" [VIM3, 4.1].

The solution proposed in [2] is to more sharply distinguish between quantities (e.g., 'indication') and quantity values (e.g., 'indication value'), whereas, on the contrary, someone may contend on the justification of the distinction itself between 'quantity' and 'quantity value'. We argue that the issue is not only a lexical one, and that it signals instead the need of a better understanding of such concepts and their relations. A proposal to this goal is presented here.

2. THE PROBLEM

Measurement is aimed at attributing quantity values to quantities [1]. It is not amazing then that the appropriate characterization of such concepts, and therefore of the related conditions of measurability [3, 4], is considered a basic task for measurement science. On the other hand, both 'quantity' and 'quantity value', as defined by the VIM3, are not entirely exempt from ambiguity.

2.1. 'Quantity'

The same term "quantity" unfortunately designates both *general* entities (such as length) and *individual* entities (such as the length of this table) (for the latter concept the three editions of the VIM have alternatively used "specific quantity" (VIM1 [5]), "particular quantity" (VIM2 [6]), and "individual quantity" (VIM3), while never defining the concept as such).

These two meanings are significantly different: while the concepts around 'system of quantities' (including 'base quantity' and 'quantity dimension') and those related to quantity types (e.g., 'ordinal quantity') involve general quantities, measurement is operatively concerned with individual quantities, such as measurement units and measurands. Indeed, the quantity to which "one or more quantity values […] can reasonably be attributed" [VIM3, 2.1] by means of measurement is clearly an individual quantity. In the following our attention will be devoted to individual quantities only.

2.2. 'Quantity value'

According to [7], an "intensional definition is a concise statement of what the concept is. It states the superordinate concept to concept expressed by the designation and its delimiting characteristics". For example, in the mentioned definition of 'measurement' given by the VIM3 'process' is the superordinate concept (i.e., measurement is a process) and the condition of experimentally obtaining... is the delimiting characteristic. Analogously, 'property' is defined by the VIM3 as the superordinate concept of 'quantity'. What about 'quantity value'? The evolution of its definition across the three editions of the VIM is a troubled one:

"the expression of a quantity in terms of a number and an appropriate unit of measurement" (VIM1);

"magnitude of a particular quantity generally expressed as a unit of measurement multiplied by a number" (VIM2);

"number and reference together expressing magnitude of a quantity" (VIM3).

Is a quantity value a linguistic entity (an "expression", as in the VIM1) or a non-linguistic one (a "magnitude", as in the VIM2)? As for the VIM3, the superordinate concept is not stated, against the mentioned condition [7], and is not clear, also because of the introduction of the (undefined) concept of reference, adopted in the VIM3 plausibly to generalize the one of unit of measurement. Indeed, according to [VIM3, 1.19 Note 1] the reference in a quantity value can be either a measurement unit, or a reference to a measurement procedure (i.e., a reference to a description), or a reference material. A definition of 'reference' encompassing these options (a quantity, a reference to a linguistic entity, a physical entity) is hard to imagine. The problem is further highlighted by considering the basic formula of quantity calculus [8]:

$$Q = \{Q\} \cdot [Q]$$

where $\{Q\}$ is a numerical quantity value and $[Q]$ is a measurement unit.

With respect to the subject of the present analysis, even the interpretation of this formula is controversial. According to Maxwell (1873, quoted in [8]), "every expression of a quantity consists of two factors...", and therefore either the left hand term is "the expression of a quantity" or the "=" symbol must stand for "is expressed by" instead of "is equal to". As just quoted, the position of the VIM3 is not the same: the "two factors" express the magnitude of a quantity, not the quantity. To make the situation even more confused, the French version of the VIM3 states that a quantity values is the "ensemble d'un nombre et d'une référence constituant l'expression quantitative d'une grandeur" ("set of a number and a reference constituting the quantitative expression of a quantity"), i.e., mirrors the Maxwell's definition, against the English version of the VIM3. All these seem to be explicit hints that the concepts '(individual) quantity' and 'quantity value' require some clarification.

3. THE PROPOSAL

Let us consider again the formula:

$$Q = \{Q\} \cdot [Q]$$

This formula admits a twofold interpretation:

- a *theoretical* interpretation, according to which the quantity Q is equal to a multiple $\{Q\}$ of the unit quantity $[Q]$; hence, the symbol "·" indicates here concatenation of individual quantities, abstracting from the entities to which such quantities belongs;
- an *operational* interpretation, according to which at least in principle there exists a procedure that allows constructing an entity that is composed by properly concatenating $\{Q\}$ replicas of the standard that realizes the unit $[Q]$, and the quantity Q characterizes an object that is indistinguishable, relative to the kind of quantity of Q, from this entity; hence, the symbol "·" indicates here concatenation of individual quantities obtained by concatenating the entities to which such quantities belongs.

The link between these two interpretations is constituted by the assumption that the individual quantity that belongs to the entity that is composed by $\{Q\}$ replicas of the standard that realizes the unit $[Q]$ is precisely $\{Q\} \cdot [Q]$.

The inverse formula:

$$\{Q\} = Q/[Q]$$

(see [VIM3, 1.20 Note 2]) further shows that it is assumed that quantities and measurement units can be in some sense divided with each other and that the result is a number.

This is the basis for the following two theses.

Thesis 1: the entities involved in the left hand side and in the right hand side of the formula $Q = \{Q\} \cdot [Q]$ are of the same sort; in particular, *both are individual quantities*.

Still, the concept of individual quantity is a complex one, as it includes both measurands, such as the length of this table, and quantity values, such as 1,23 m:

- the first sort of individual quantities (e.g., the length of this table) includes quantities that are typically specified by pointing out entities that instantiate them, such as this table; hence, such quantities may be called *concrete individual quantities*, since they are specified by reference to a concrete entity, that can be called their *address*;
- the second sort of individual quantities (e.g., 1,23 m) includes quantities that are specified by referring to a given element of a set of individual quantities determined by the choice of a unit (in the example the element is 1,23 and the unit is m); such quantities may be called *abstract individual quantities*, since they are specified by reference to an abstract entity, that can be called their *class* within a given *classification*.

Measurands are typically concrete individual quantities, while quantity values are abstract individual quantities. Furthermore, a value is always an element of a set of values, i.e., a class within a given classification. As a consequence, the formula $Q = \{Q\} \cdot [Q]$ can be understood as follows:

1) Q is a *concrete individual quantity*

2) $\{Q\} \cdot [Q]$ is a specific *abstract individual quantity*, where:

2.1) $[Q]$ determines a classification, i.e., a set of distinguishable elements

2.2) $\{Q\}$ identifies a class within the classification determined by $[Q]$, i.e., $\{Q\}$ is the *identifier* of a class.

This concrete / abstract duality can be then used to clarify the meaning of the "=" symbol in the above formula, and therefore both the structure of the process of measurement and the way in which such process provides information concerning the system under measurement:

Concrete individual quantities,	*Abstract individual quantities*,
such as the length of this table,	such as 1,23 m,
are elements of the world,	are elements of a classification,
are unknown before measurement,	are assumed to be known before measurement,
are individuated in terms of a given system under measurement	are individuated independently of any system under measurement
as measurands	as quantity values
that are represented by quantity values.	that are assumed to represent measurands.

Hence:

Thesis 2: *there is a duality between concrete and abstract individual quantities*.

This framework gives a basic justification of the epistemic role of measurement, thus interpreted as a process in which the concrete individual quantity of a given entity, which is known by address, in as much as it is empirically given, turns out to be known also by description, i.e., by reference to a class in a given classification.

Accordingly, measurement is modeled as involving four stages:

- stage 1 – *definition of the measurand*:
  $\mathrm{Q} = \mathrm{Q}(x)$

the individual quantity Q is specified as the quantity $\mathrm{Q}(x)$ of a given system x;

- stage 2 – *calibration*:
- $\mathrm{Q}(u) = [\mathrm{Q}]$

a given system u is recognized as the standard that realizes the unit [Q], i.e., the quantity $\mathrm{Q}(u) = 1 \cdot [\mathrm{Q}]$;

- stage 3 – *experimental comparison*:
  $\mathrm{Q}(x) \approx \{\mathrm{Q}\} \cdot \mathrm{Q}(u)$

$\mathrm{Q}(x)$ is recognized as indistinguishable from the quantity of a sequence of {Q} replicas of the system u;

- stage 4 – *knowledge of the measurand*:
  $\mathrm{Q} = \{\mathrm{Q}\} \cdot [\mathrm{Q}]$

the individual quantity Q is known to be equal to $\{\mathrm{Q}\} \cdot [\mathrm{Q}]$.

These four stages emphasize the conceptual aspects that critically characterize a measurement, that is aimed at complementing the knowledge on the measurand available before measurement (stage 1) by assigning it to a class $\{\mathrm{Q}\} \cdot [\mathrm{Q}]$ of a given classification [Q] (stage 4), where the assignment results from two experimental sub-processes, for calibrating the system that generates the classification (stage 2) and comparing it with the system under measurement (stage 3). The framework also highlights the inter-subjective role of measurement, guaranteed by the fact that the information it provides can be communicated as an abstract individual quantity: indeed, if a measurand cannot always be shared, a classification can always be.

4. A GENERALIZATION

The proposed interpretation of the relation between quantities and quantity values is independent of the algebraic structure (i.e., on the "scale type") of the involved entities: what has been considered in terms of quantities and quantity values can be immediately generalized to generic properties (as mentioned, 'property' is the superordinate concept of 'quantity') and property values, thus extending its scope. As a straightforward analogous of $\mathrm{Q} = \{\mathrm{Q}\} \cdot [\mathrm{Q}]$, the following formula can be introduced:

$$\mathrm{P} = \{\mathrm{P}\} \text{ in } [\mathrm{P}]$$

where P is an individual concrete property (e.g., the color of the surface of this object), [P] is a P-related classification (e.g., [red, orange, yellow, green, blue, violet]), and {P} is a class in [P] (e.g., blue), so that the formula is read, e.g., the color of the surface of this object is blue in (the classification) [red, orange, …] (when the classification is presented as a scale, the property value is sometimes reported as "{P} on [P]", as in the case "the hardness of graphite is 1,5 on the Mohs scale"). The related inverse formula:

$$\{\mathrm{P}\} = \mathrm{P} \text{ in } [\mathrm{P}]$$

can be interpreted in the same, unproblematic way (e.g., "3 is the wind speed here now on the Beaufort scale").

The previous analysis about $\mathrm{Q} = \{\mathrm{Q}\} \cdot [\mathrm{Q}]$ can now be generalized to $\mathrm{P} = \{\mathrm{P}\}$ in [P]. In particular, on the operational side, it is assumed here that at least in principle there exists a procedure that allows generating an entity that is an element of the class {P} in classification [P], and that this entity is indistinguishable by comparison from the entity to which P belongs, so that the individual property P is precisely {P} in [P]. This conclusion shows that numbers do not play an essential role in this conceptualization.

5. CONCLUSIONS

The theses presented here propose a simple and consistent conceptualization for 'quantity' and 'quantity value', and actually, and more generally, for 'property' and 'property value'. They provide: i) a distinction between abstract and concrete individual quantities, grounding the twofold interpretation of the basic formula Q = {Q}·[Q]; ii) an unequivocal definition of the concept 'quantity value'; iii) a generalization of these concepts to properties and property values; iv) an insight into the conceptual structure of measurement as a process aimed at exploiting the duality between abstract and concrete individual quantities / properties, by merging the information available on the measurand before its measurement with the information obtained experimentally.

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